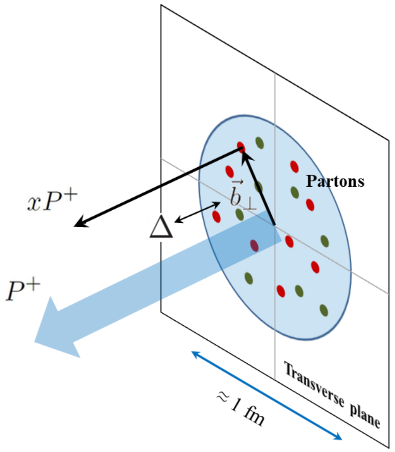
CNF lecture series, July13-17



Hadron structure with GPDs (2/5)

Cédric Lorcé

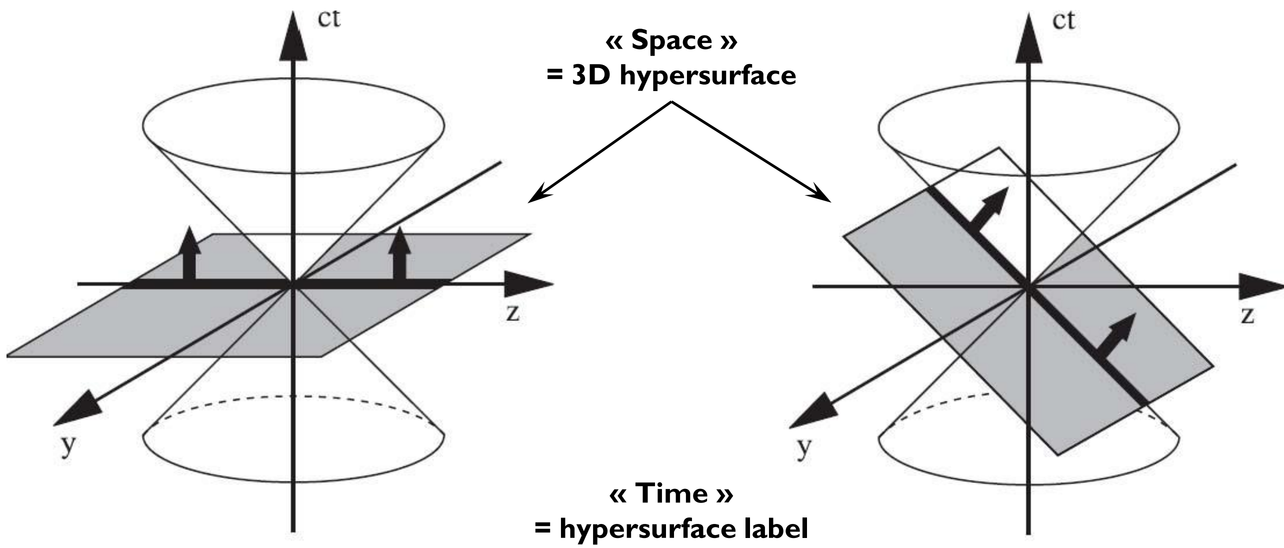
Light-front picture

Forms of dynamics

Space-time foliation

Light-front components

$$a^{\pm} = \frac{1}{\sqrt{2}}(a^0 \pm a^3)$$

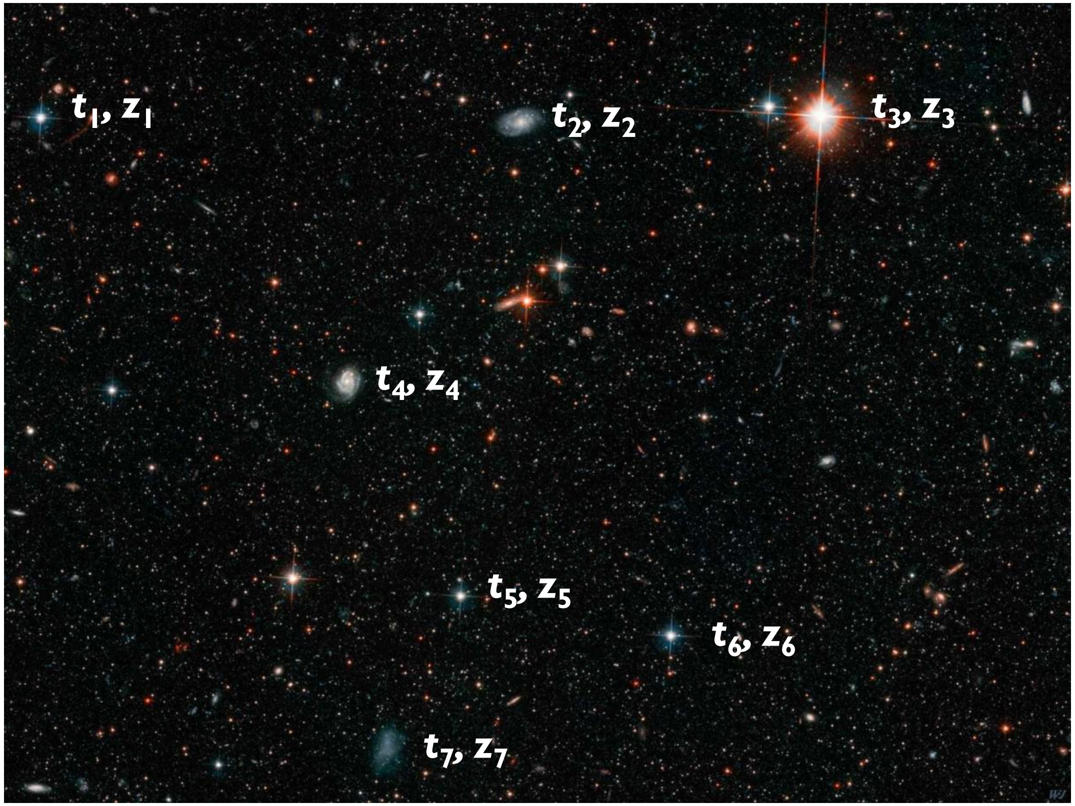


	Instant form dynamics	**Light-front form dynamics**
Time	x^0	x^+
Space	$\vec{x}$	$\vec{x}_\perp, x^-$
Energy	p^0	p^-
Momentum	$\vec{p}$	$\vec{p}_\perp, p^+$

[Dirac, RMP21 (1949) 292]
[Brodsky, Pauli, Pinsky, PR301 (1998) 299]

Instant form *vs* light-front form

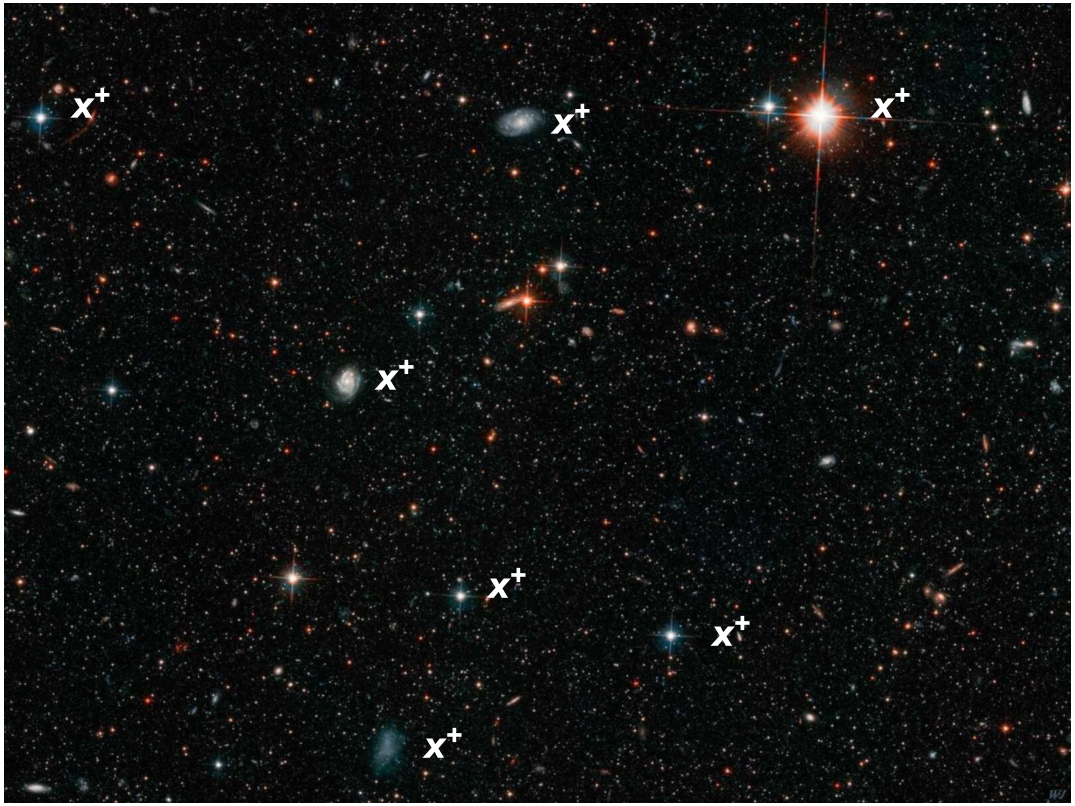
Ordinary point of view



The farther away, the earlier

Instant form *vs* light-front form

Light front point of view $x^+ = \frac{1}{\sqrt{2}}(t+z)$

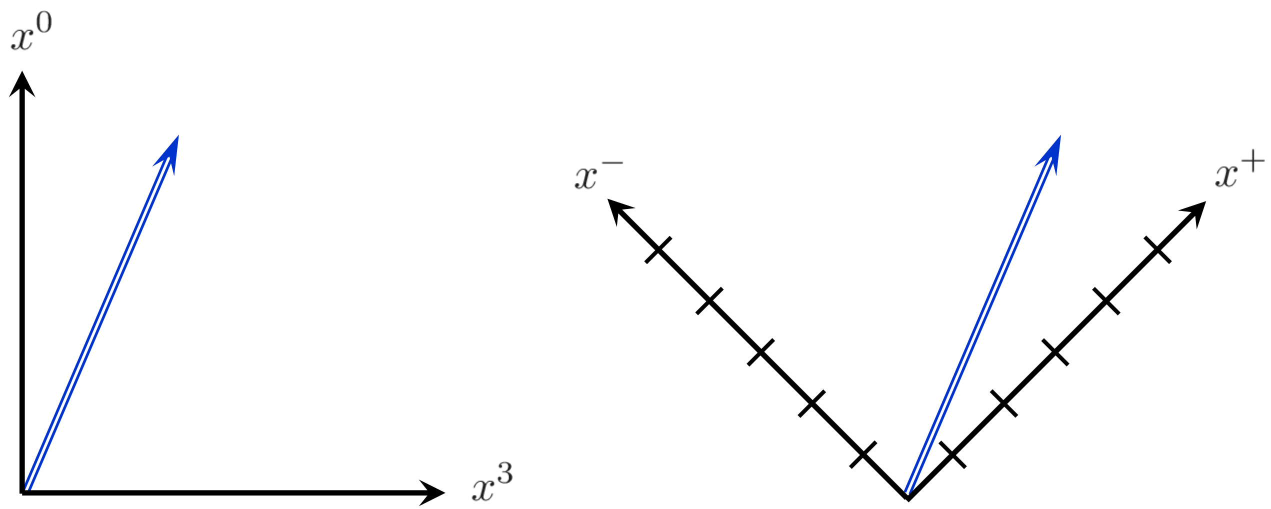


Change in synchronization protocol !

[Freese, Miller, PRD107 (2023) 7, 074036]
[Freese, Miller, PRD108 (2023) 9, 094026]

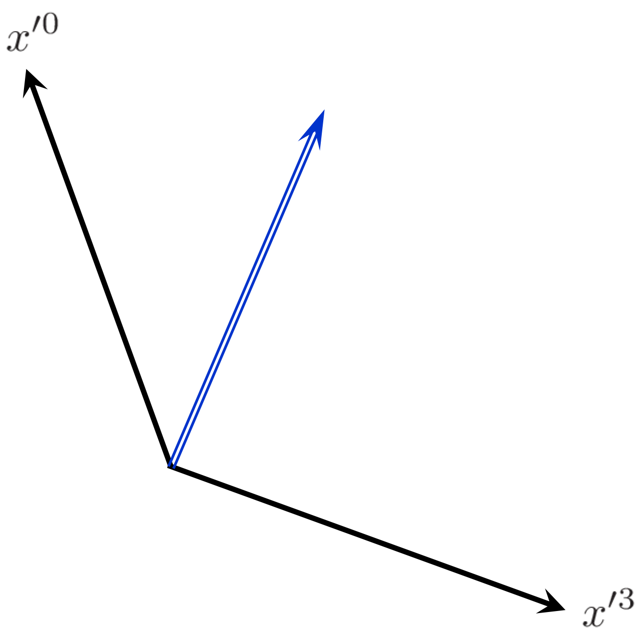
Instant form *vs* light-front form

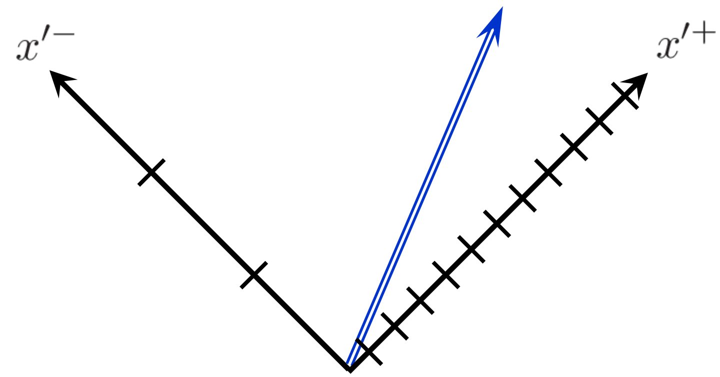
Initial frame



Instant form *vs* light-front form

Boosted frame





$$x'^0 = \frac{x^0 + \beta x^3}{\sqrt{1-\beta^2}}$$

$$x'^3 = \frac{x^3 + \beta x^0}{\sqrt{1-\beta^2}}$$

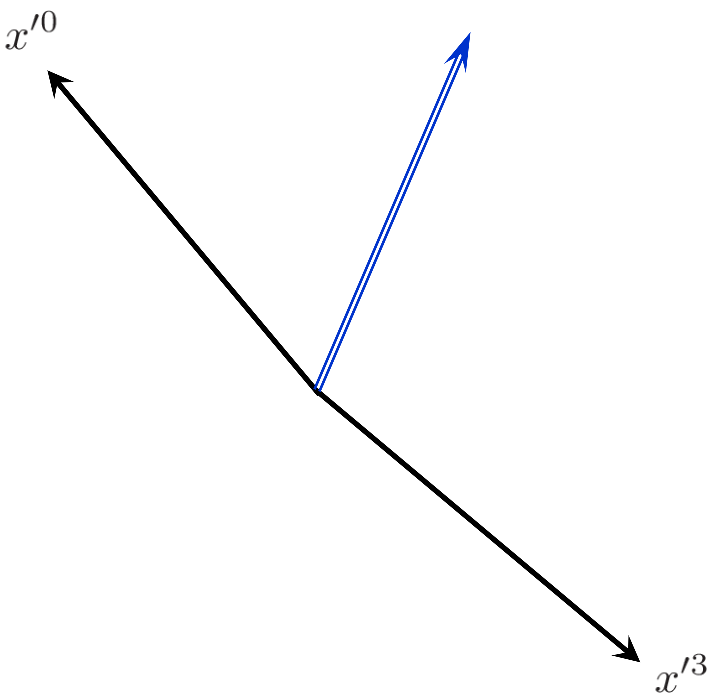
$$x'^+ = \sqrt{\frac{1+\beta}{1-\beta}}\, x^+ = e^{\eta}\, x^+$$

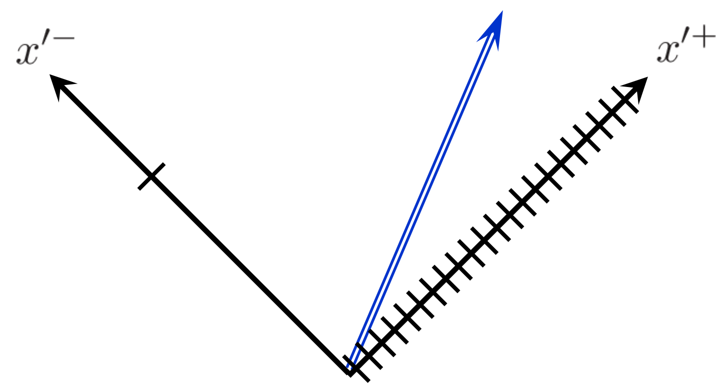
$$x'^- = \sqrt{\frac{1-\beta}{1+\beta}}\, x^- = e^{-\eta}\, x^-$$

Relativistic (longitudinal) Doppler effect !

Instant form *vs* light-front form

Infinite-momentum frame $\beta \to 1$





$$x'^0 \approx x'^3 \approx \frac{x'^+}{\sqrt{2}}$$

$$x'^- \approx 0$$

Galilean/non-relativistic algebra

Galilean symmetry

$$[J^i, J^j] = i\epsilon^{ijk} J^k \qquad [B^i, P^j] = -i\delta^{ij} M$$

$$[J^i, B^j] = i\epsilon^{ijk} B^k \qquad [B^i, H] = -iP^i$$

$$[B^i, B^j] = 0 \qquad [B^i, M] = [J^i, M] = [J^i, H] = 0$$

Position operator (center of inertia)

$$\boxed{B^i = -MR^i} \Rightarrow \begin{aligned} &[R^i, P^j] = i\delta^{ij}\mathbb{1} \\ &[R^i, R^j] = 0 \\ &[J^i, R^j] = i\epsilon^{ijk} R^k \end{aligned}$$

Galilean boost

$$M' = M \qquad \vec{p}\,' = \vec{p} + M\vec{v}$$

[Susskind, PR165 (1968) 1535]
[Kogut, Soper, PRD1 (1970) 2901]
[Burkardt, IJMPA18 (2003) 2, 173]
[C.L., EPJC78 (2018) 9, 785]

Light-front Poincaré algebra

$$B_\perp^1 = \tfrac{1}{\sqrt{2}}(K^1 + J^2) \qquad \mathcal{J}_\perp^1 = \tfrac{1}{\sqrt{2}}(J^1 + K^2)$$
$$B_\perp^2 = \tfrac{1}{\sqrt{2}}(K^2 - J^1) \qquad \mathcal{J}_\perp^2 = \tfrac{1}{\sqrt{2}}(J^2 - K^1)$$

Transverse space-time symmetry

$$[J^3, \mathcal{J}_\perp^i] = i\epsilon^{3ij}\mathcal{J}_\perp^j \qquad [B_\perp^i, P_\perp^j] = -i\delta_\perp^{ij}P^+$$
$$[J^3, B_\perp^i] = i\epsilon^{3ij}B_\perp^j \qquad [B_\perp^i, P^-] = -iP_\perp^i$$
$$[B_\perp^i, B_\perp^j] = 0 \qquad [B_\perp^i, P^+] = [J^3, P^+] = [J^3, P^-] = 0$$

Transverse position operator (center of P^+)

$$\boxed{B_\perp^i = -P^+ R_\perp^i} \quad \Rightarrow \quad \begin{aligned} &[R_\perp^i, P_\perp^j] = i\delta_\perp^{ij}\mathbb{1} \\ &[R_\perp^i, R_\perp^j] = 0 \\ &[J^3, R_\perp^i] = i\epsilon^{3ij}R_\perp^j \end{aligned}$$

Transverse boost

$$p'^+ = p^+ \qquad \vec{p}\,'_\perp = \vec{p}_\perp + p^+\vec{v}_\perp$$

[Susskind, PR165 (1968) 1535]
[Kogut, Soper, PRD1 (1970) 2901]
[Burkardt, IJMPA18 (2003) 2, 173]
[C.L., EPJC78 (2018) 9, 785]

Light-front densities

Normalizations $\langle p'^+, \vec{p}'_\perp | p^+, \vec{p}_\perp \rangle = 2p^+ (2\pi)^3 \, \delta(p'^+ - p^+) \, \delta^{(2)}(\vec{p}'_\perp - \vec{p}_\perp)$

$\langle p'^+, \vec{r}'_\perp | p^+, \vec{r}_\perp \rangle = 2p^+ \, 2\pi \, \delta(p'^+ - p^+) \, \delta^{(2)}(\vec{r}'_\perp - \vec{r}_\perp)$

Localized states

momentum space $|p^+, \vec{p}_\perp\rangle$

mixed space $$|p^+, \vec{r}_\perp\rangle = \int \frac{\mathrm{d}^2 p_\perp}{(2\pi)^2} \, e^{-i\vec{p}_\perp \cdot \vec{r}_\perp} \, |p^+, \vec{p}_\perp\rangle$$

Impact-parameter dependent distributions (IPDs) $x^+ = 0$

$$\langle p'^+, \vec{r}'_\perp | \int \mathrm{d}x^- \, J^+(x) | p^+, \vec{r}_\perp \rangle = 2p^+ \, 2\pi \, \delta(p'^+ - p^+) \, \delta^{(2)}(\vec{r}'_\perp - \vec{r}_\perp)$$

$$\times \underbrace{\int \frac{\mathrm{d}^2 \Delta_\perp}{(2\pi)^2} \, e^{-i\vec{\Delta}_\perp \cdot (\vec{x}_\perp - \vec{r}_\perp)} \, \frac{\langle P^+, \frac{\vec{\Delta}_\perp}{2} | J^+(0) | P^+, -\frac{\vec{\Delta}_\perp}{2} \rangle}{2P^+}}_{\text{Internal distribution}}$$

Internal distribution

NB: $\int \mathrm{d}x^- \Leftrightarrow \Delta^+ = 0$ (Drell-Yan frame) **is essential to ensure probabilistic interpretation !**

[Soper, PRD15 (1977) 1141]
[Burkardt, PRD62 (2000) 071503]
[Diehl, EPJC25 (2002) 223]
[Burkardt, IJMPA18 (2003) 2, 173]

Light-front densities

Unpolarized quark IPD $\vec{b}_\perp = \vec{x}_\perp - \vec{r}_\perp$

$$q(x, \vec{b}_\perp) = \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2}\, e^{-i\vec{\Delta}_\perp\cdot\vec{b}_\perp}\, H_q(x, 0, -\vec{\Delta}^2_\perp)$$

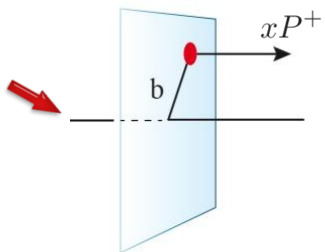
$$H_q(x,\xi,t)$$

$$\xi = -\frac{\Delta^+}{2P^+}$$

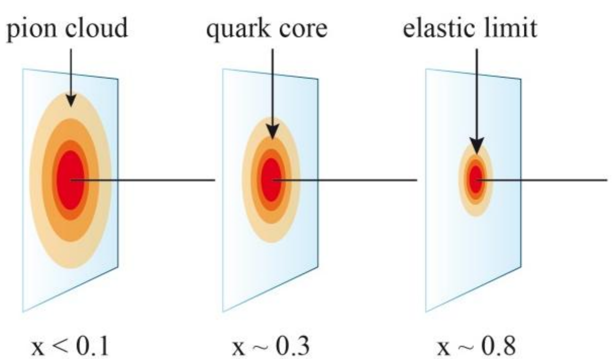
$$t = \Delta^2 = 2\Delta^+\Delta^- - \vec{\Delta}^2_\perp$$

Center of P^+ (light-front inertia) $B^i_\perp = M^{+i} = \int \mathrm{d}x^-\,\mathrm{d}^2x_\perp\,(x^+T^{+i} - x^i_\perp T^{++})$

$$\begin{aligned}\vec{R}_\perp &= \frac{1}{P^+}\int \mathrm{d}x^-\,\mathrm{d}^2x_\perp\,\vec{x}_\perp\,T^{++}(x)\\ &= \frac{1}{P^+}\sum_n \vec{r}_{\perp,n}\,p^+_n = \sum_n x_n\,\vec{r}_{\perp,n}\end{aligned}$$



$$x \equiv \frac{p^+}{P^+}$$

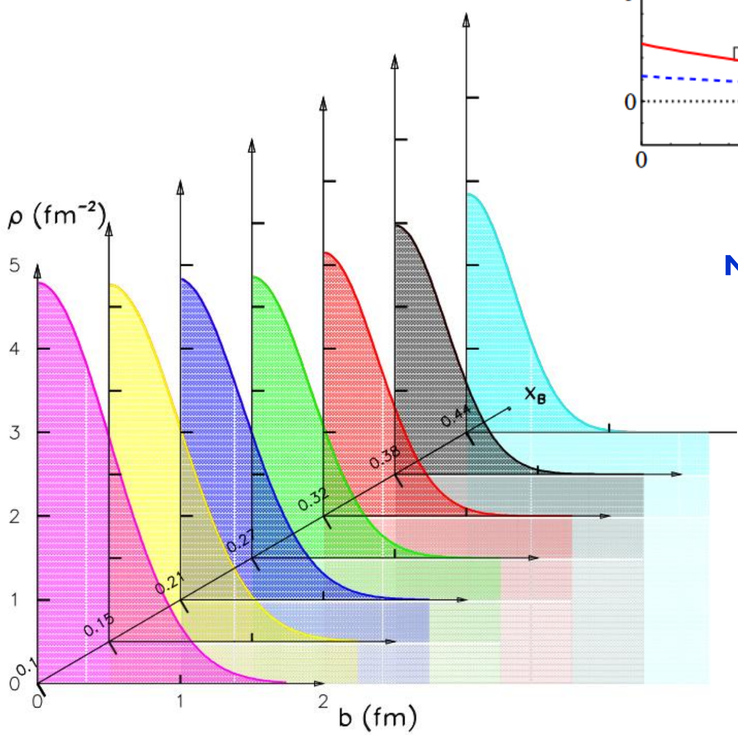


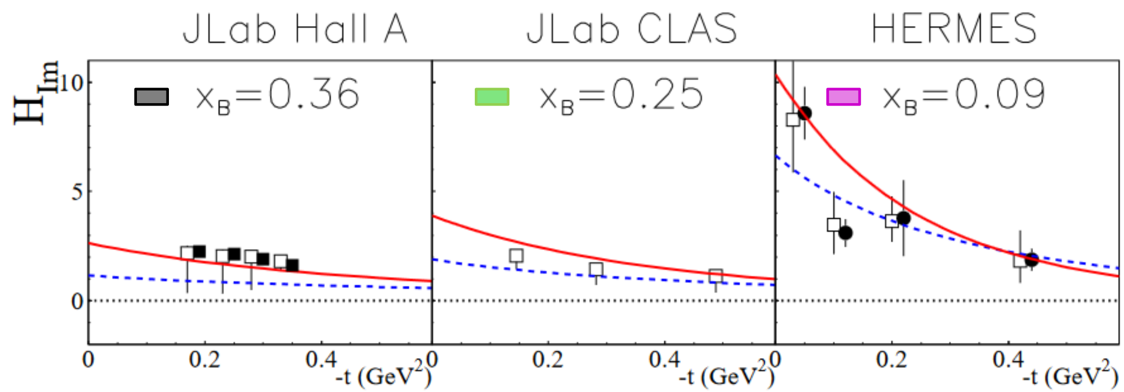
$$\langle\vec{b}^2_\perp\rangle(x) = \frac{\int \mathrm{d}^2b_\perp\,\vec{b}^2_\perp\,q(x,\vec{b}_\perp)}{\int \mathrm{d}^2b_\perp\,q(x,\vec{b}_\perp)} \xrightarrow{x\to 1} 0$$

as a result of $x_n \geq 0, \quad \sum_n x_n = 1$

[Burkardt, IJMPA18 (2003) 2, 173]

Light-front densities





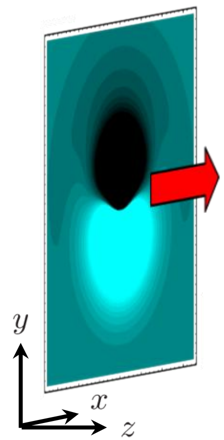
Flat in *t*
Narrow in *b*

Steep in *t*
Wide in *b*

[Guidal, Moutarde Vanderhaeghen, RPP76 (2013) 066202]

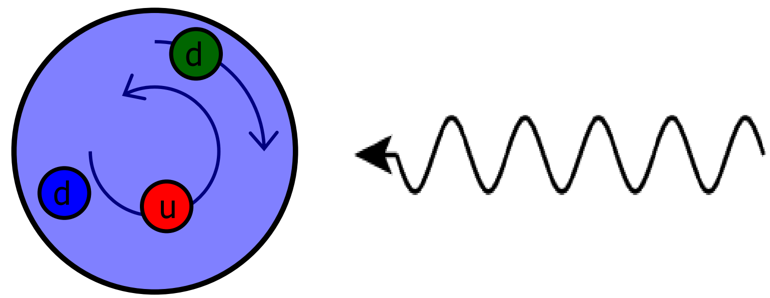
Light-front artifacts

$$J^+ = \frac{J^0 + J^3}{\sqrt{2}}$$



Neutron at rest

$J^3_{\text{rest}} < 0$



$J^3_{\text{rest}} > 0$

$\vec{S} \otimes$ $\qquad \mu_n = -1.91$

Magnetic dipole moment

$$\vec{E}'_\perp = \gamma\,(\vec{E}_\perp + \vec{v} \times \vec{B}_\perp) \;\Rightarrow\; \vec{d}'_\perp(\vec{r}\,') = \gamma\,\vec{v} \times \vec{\mu}_\perp(\vec{r})$$

$$\boxed{\vec{d}'_\perp = \vec{v} \times \vec{\mu}_\perp} \qquad \int \mathrm{d}^3 r' = \int \frac{\mathrm{d}^3 r}{\gamma}$$

Induced electric dipole moment

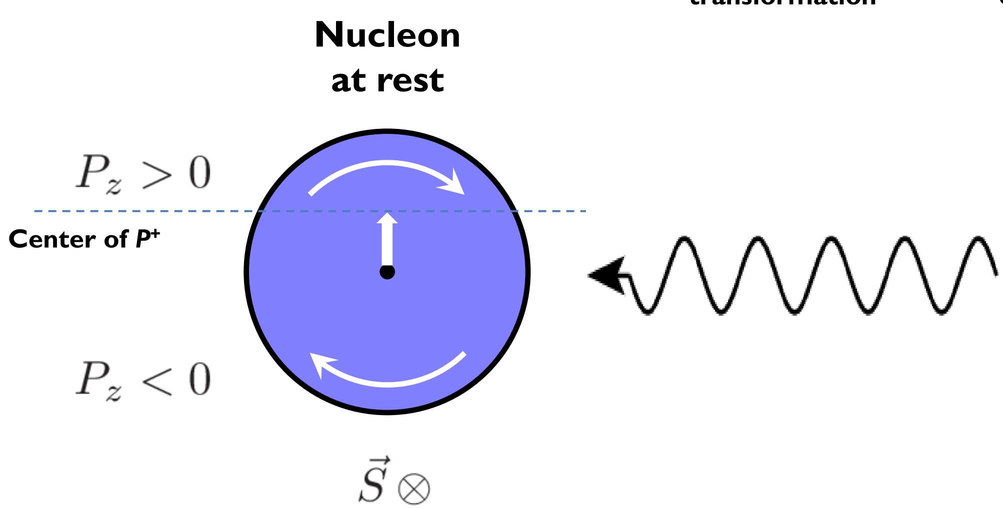
[Burkardt, IJMPA18 (2003) 2, 173]
[Carlson, Vanderhaeghen, PRL100 (2008) 032004]

Light-front artifacts

Transverse electric dipole moment $\hat{S} = \vec{S}/|\vec{S}|$ $\vec{e}_z \Leftrightarrow \vec{v}_{\rm IMF}$

$$\boxed{\langle \vec{d}_\perp \rangle = \int \mathrm{d}^2 b_\perp\, \vec{b}_\perp\, \rho_{\rm LF}(\vec{b}_\perp) = \frac{\vec{e}_z \times \hat{S}_\perp}{2M_N}\,\kappa_N}$$

$$= \underbrace{\frac{\vec{e}_z \times \hat{S}_\perp}{2M_N} G_M(0)}_{\text{Expected from Lorentz transformation}} - \underbrace{\frac{\vec{e}_z \times \hat{S}_\perp}{2M_N}}_{\text{Transverse shift of center of } P^+} \overbrace{G_E(0)}^{\text{Electric charge}}$$



[Burkardt, IJMPA18 (2003) 2, 173]
[C.L., EPJC78 (2018) 785]
[Chen, C.L., PRD107 (2023) 9, 096003]

Phase-space picture

Position operator in QFT

Textbook claim: « R^μ **does not exist** »

In reality… it depends on what you mean or want !

Center	Position operator	Canonical relation $[R^i, P^j] = i\delta^{ij}$	Vector under rotations $[J^i, R^j] = i\epsilon^{ijk} R^k$	Compatibility of components $[R^i, R^j] = 0$	Four-vector transformation $R'^\mu = \Lambda^\mu{}_\nu R^\nu$	
***P*⁺**	$R^i_\perp = \frac{1}{P^+}\int \mathrm{d}x^-\,\mathrm{d}^2x_\perp\, x^i_\perp T^{++} \overset{x^+=0}{=} -\frac{B^i_\perp}{P^+}$	✓	✓	✓	✗	2D
Energy	$R^i_E = \frac{1}{P^0}\int \mathrm{d}^3x\, x^i T^{00} \overset{x^0=0}{=} -\frac{K^i}{P^0}$	✓	✓	✗	✗	
Mass	$R^\mu_M = \Lambda^\mu{}_\nu R^\nu_E\big\vert_{\text{rest}}$	✓	✓	✗	✓	3D
Canonical (or spin)	$R^i_c = \frac{P^0 R^i_E + M R^i_M}{P^0 + M}$	✓	✓	✓	✗	

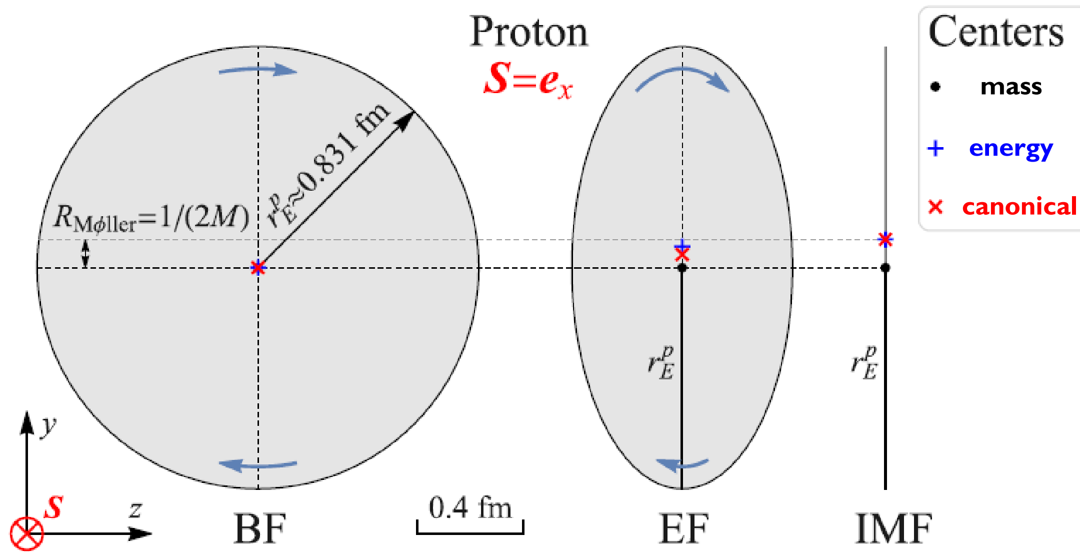
→ **Localized states on spacelike hypersurface** $x^0 = 0$

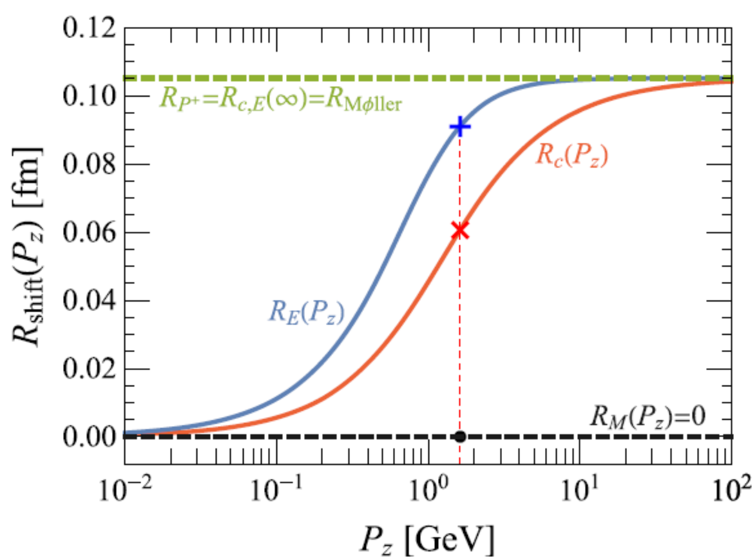
$$\vec{R}_c\,|\vec{r}\rangle = \vec{r}\,|\vec{r}\rangle$$

$$\boxed{|\vec{r}\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2p^0}}\, e^{-i\vec{p}\cdot\vec{r}}\,|p\rangle}$$

[Pryce, PRSLA195 (1948) 62]
[Newton, Wigner, RMP21 (1949) 3, 400]
[Fleming, PR137 (1965) B188]
[C.L., EPJC78 (2018) 785]

Position operator in QFT





[C.L., EPJC78 (2018) 785]
[Chen, C.L., PRD107 (2023) 9, 096003]

Spatial distributions (general formalism)

Phase-space representation

$$\langle\psi|O(x)|\psi\rangle = \int \frac{\mathrm{d}^3P}{(2\pi)^3}\,\mathrm{d}^3R\,\rho_\psi(\vec{R},\vec{P})\,\langle O\rangle_{\vec{R},\vec{P}}(x)$$

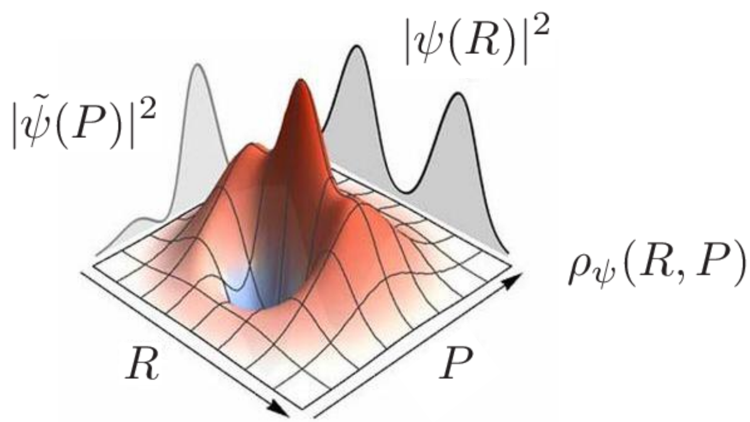
Nucleon Wigner distribution

$$\rho_\psi(\vec{R},\vec{P}) = \int \mathrm{d}^3z\,e^{-i\vec{P}\cdot\vec{z}}\,\psi^*(\vec{R}-\tfrac{\vec{z}}{2})\psi(\vec{R}+\tfrac{\vec{z}}{2})$$
$$= \int \frac{\mathrm{d}^3q}{(2\pi)^3}\,e^{-i\vec{q}\cdot\vec{R}}\,\tilde{\psi}^*(\vec{P}+\tfrac{\vec{q}}{2})\tilde{\psi}(\vec{P}-\tfrac{\vec{q}}{2})$$

$$\psi(\vec{r}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,e^{i\vec{p}\cdot\vec{r}}\,\tilde{\psi}(\vec{p})$$
$$\tilde{\psi}(\vec{p}) = \frac{\langle p|\psi\rangle}{\sqrt{2p^0}}$$

Quasi-probabilistic interpretation

$$\int \mathrm{d}^3R\,\rho_\psi(\vec{R},\vec{P}) = |\tilde{\psi}(\vec{P})|^2$$
$$\int \frac{\mathrm{d}^3P}{(2\pi)^3}\,\rho_\psi(\vec{R},\vec{P}) = |\psi(\vec{R})|^2$$



[Wigner, PR40 (1932) 749]
[Carruthers, Zachariasen, PRD13 (1976) 4, 950]
[Hillery, O'Connell, Scully, Wigner, PR106 (1984) 121]
[Bialynicki-Birula, Gornicki, Rafelski, PRD 44 (1991) 1825]

Relativistic spatial distributions

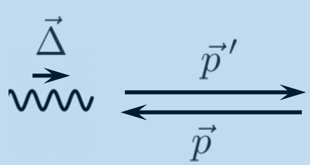
Internal distribution (for a state « localized » in phase-space) $x^0 = 0$

$$\langle O\rangle_{\vec{R},\vec{P}}(\vec{x}) = \int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{-i\vec{\Delta}\cdot(\vec{x}-\vec{R})}\, \frac{\langle P+\frac{\Delta}{2}|O(0)|P-\frac{\Delta}{2}\rangle}{\sqrt{4(P^0)^2-(\Delta^0)^2}}$$

$$= \langle O\rangle_{\vec{0},\vec{P}}(\vec{r}), \qquad \vec{r} = \vec{x}-\vec{R}$$

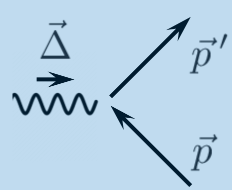
Elastic frames $\Delta^0 = \dfrac{\vec{P}\cdot\vec{\Delta}}{P^0} \stackrel{!}{=} 0$ (no energy transfer ➡ same initial and final boost factor)

$|\vec{P}| = 0$

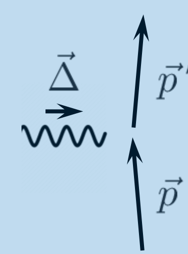


BF

$|\vec{P}| \neq 0$



$|\vec{P}| \gg M$



IMF

[C.L., Mantovani, Pasquini, PLB776 (2018) 38]
[C.L., EPJC78 (2018) 9, 785]
[C.L., Moutarde, Trawinski, EPJC79 (2019) 89]

Relativistic spatial distributions

Fundamental features

1) **The notion of spatial distribution relies on simultaneity**
2) **Probabilistic interpretation requires factorization of $\vec{P}$-dependence**

Condition 2) is satisfied by Galilean symmetry

… but not Lorentz symmetry ! $P^0 = \sqrt{\vec{P}^2 + M_N^2(1+\tau)}$

***Traditional perspective*: maintain strict probabilistic interpretation by**

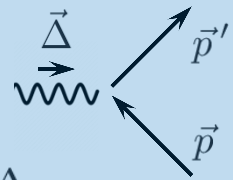
- **either neglecting recoil corrections** (Sachs approach)
- **or restricting to Galilean subgroup** (IMF/LF approach)

***Phase-space perspective:* relax to quasi-probabilistic interpretation but fully account for frame dependence !**

[C.L., EPJC78 (2018) 9, 785]
[C.L., PRL125 (2020) 232002]
[Freese, Miller, PRD105 (2022) 1, 014024]
[Freese, Miller, PRD108 (2023) 3, 034008]

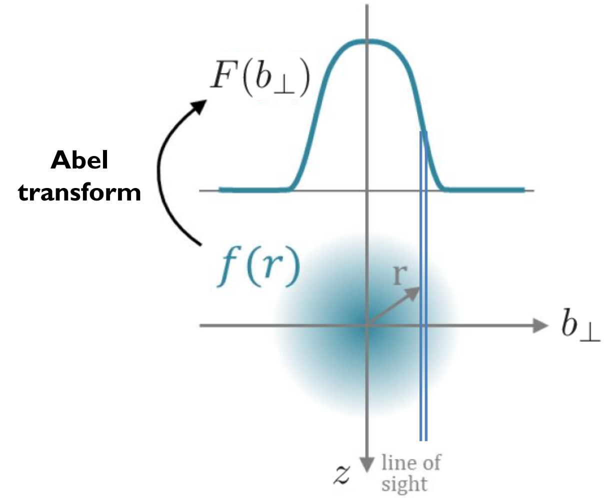
Relativistic spatial distributions

Elastic frame



$$\vec{P} = P_z \vec{e}_z \quad \Rightarrow \quad \Delta^0 = \frac{P_z \Delta_z}{P^0}$$

$$\Delta^0 = 0 \quad \Rightarrow \quad \Delta_z = 0 \quad \Leftrightarrow \quad \int \mathrm{d}z$$



2D charge distribution

$$\boxed{\begin{aligned} \rho_E^{\mathrm{EF}}(\vec{b}_\perp; P_z) &\equiv \int \mathrm{d}z\, \langle J^0 \rangle_{\vec{R}, P_z \vec{e}_z}(\vec{x}) \\ &= \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2}\, e^{-i\vec{\Delta}_\perp \cdot \vec{b}_\perp}\, \left.\frac{\langle p', s' | J^0(0) | p, s \rangle}{2P^0}\right|_{\mathrm{EF}} \end{aligned}}$$

Interpolates between BF and IMF

$$\rho_E^{\mathrm{EF}}(\vec{b}_\perp; 0) = \int \mathrm{d}z\, \rho_E^{\mathrm{BF}}(\vec{r})$$

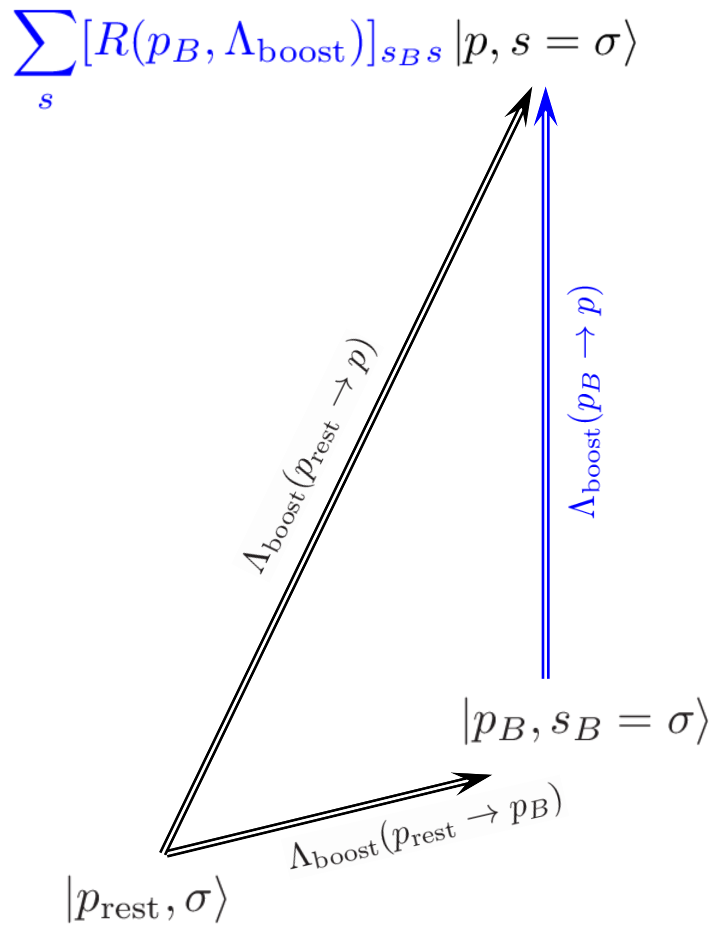
$$\vec{b}_\perp = \vec{x}_\perp - \vec{R}_\perp$$

$$\rho_E^{\mathrm{EF}}(\vec{b}_\perp; \infty) = \rho_E^{\mathrm{IMF}}(\vec{b}_\perp)$$

[C.L., Mantovani, Pasquini, PLB776 (2018) 38]
[C.L., PRL125 (2020) 232002]
[Panteleeva, Polyakov, PRD104 (2021) 1, 014008]
[Kim, Kim, PRD104 (2021) 7, 074003]

Thomas-Wigner rotation

Relativistic boosts do not commute ! $[K^i, K^j] = -i\epsilon^{ijk} J^k$



[Thomas, Nature 117 (1926) 514]
[Wigner, ZP124 (1948) 665]
[Wigner, RMP29 (1957) 255]

Four-current amplitude

Lorentz transformation of an off-forward amplitude

$$\langle p', s'|J^\mu(0)|p, s\rangle = \sum_{s'_B, s_B} D^{*(j)}_{s'_B s'}(p'_B, \Lambda) D^{(j)}_{s_B s}(p_B, \Lambda)\, \Lambda^\mu{}_\nu\, \langle p'_B, s'_B|J^\nu(0)|p_B, s_B\rangle$$

[Durand, De Celles, Marr, PR126 (1962) 1882]

Confirmed by explicit evaluation of spinors

$\hat{\Delta} = \vec{\Delta}/|\vec{\Delta}|$ $\qquad$ $Q^2|_{\rm EF} = \vec{\Delta}^2_\perp$

$$\langle p', s'|J^0(0)|p, s\rangle\big|_{\rm EF} = 2M_N\,\gamma\,\Big\{\Big[\delta_{s's}\cos\theta + (\vec{\sigma}_{s's}\times i\hat{\Delta}_\perp)_z\,\sin\theta\Big]\,G_E(Q^2) + \beta\,\Big[-\delta_{s's}\sin\theta + (\vec{\sigma}_{s's}\times i\hat{\Delta}_\perp)_z\,\cos\theta\Big]\,\sqrt{\tau}G_M(Q^2)\Big\}$$

$$\langle p', s'|J^3(0)|p, s\rangle\big|_{\rm EF} = 2M_N\,\gamma\,\Big\{\beta\,\Big[\delta_{s's}\cos\theta + (\vec{\sigma}_{s's}\times i\hat{\Delta}_\perp)_z\,\sin\theta\Big]\,G_E(Q^2) + \Big[-\delta_{s's}\sin\theta + (\vec{\sigma}_{s's}\times i\hat{\Delta}_\perp)_z\,\cos\theta\Big]\,\sqrt{\tau}G_M(Q^2)\Big\}$$

$$\langle p', s'|J^i_\perp(0)|p, s\rangle\big|_{\rm EF} = 2M_N\,(\sigma_z)_{s's}\,(\vec{e}_z\times i\hat{\Delta}_\perp)^i\,\sqrt{\tau}G_M(Q^2)$$

Boost parameters

$$\beta = \frac{P_z}{P^0} \qquad \gamma = \frac{P^0}{\sqrt{P^2}}$$

$P^2 = M_N^2(1+\tau)$

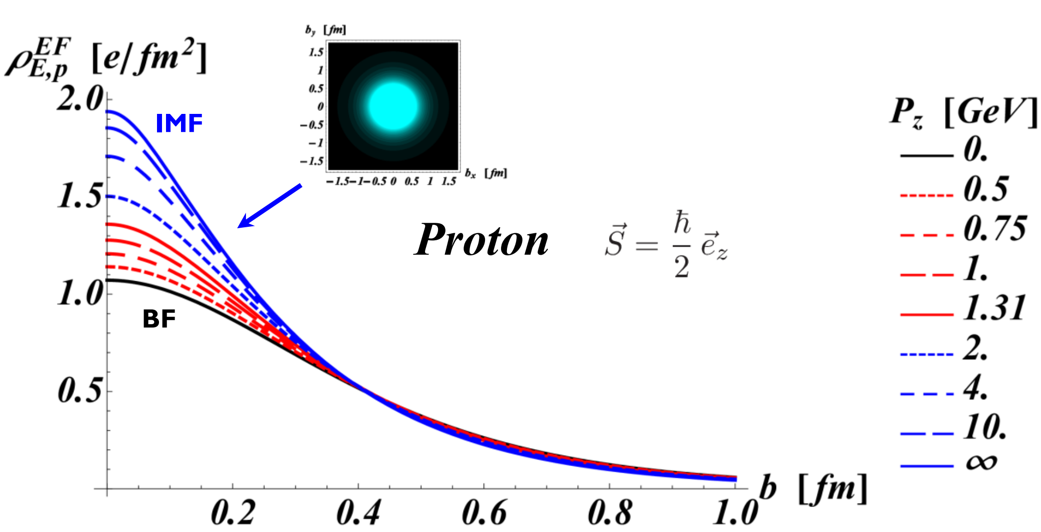
Wigner rotation

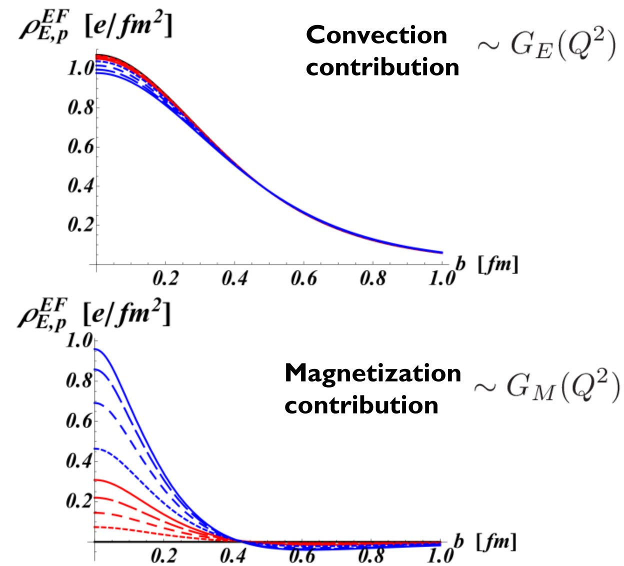
$$\cos\theta = \frac{P^0 + M_N(1+\tau)}{(P^0+M_N)\sqrt{1+\tau}}$$

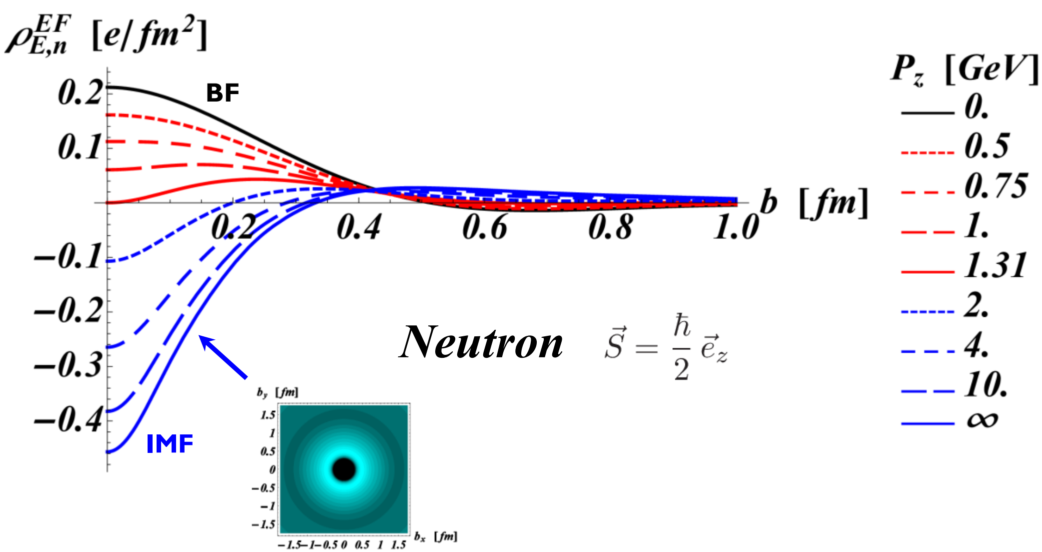
$$\sin\theta = -\frac{\sqrt{\tau}P_z}{(P^0+M_N)\sqrt{1+\tau}}$$

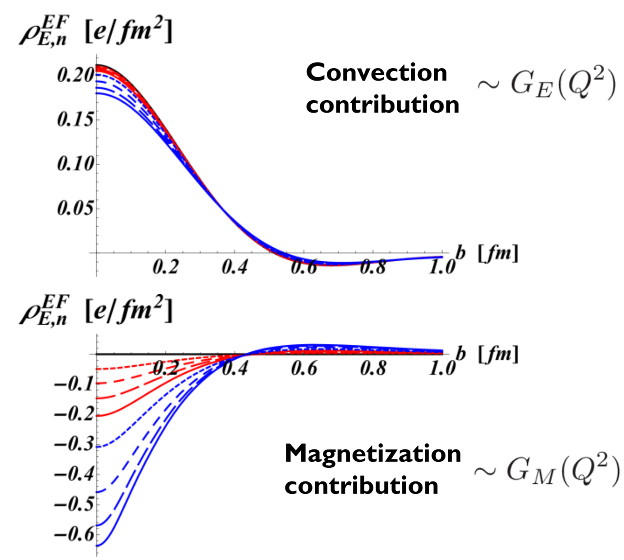
[Chung, Polyzou, Coester, Keister, PRC37 (1988) 2000]
[Rinehimer, Miller, PRC80 (2009) 015201]
[C.L., Wang, PRD105 (2022) 9, 096032]
[Chen, C.L., PRD106 (2022) 11, 116024]

EF charge distributions (longitudinal polarization)





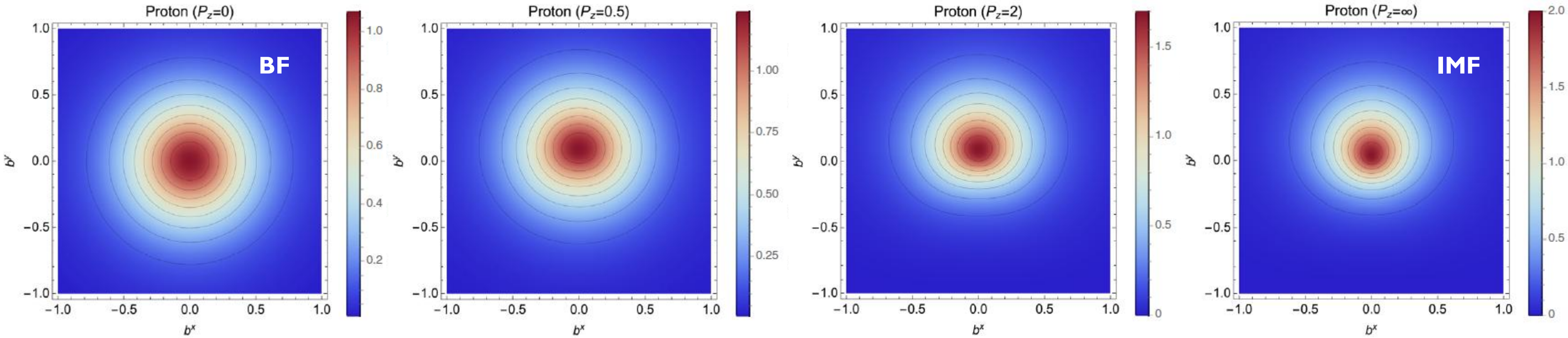




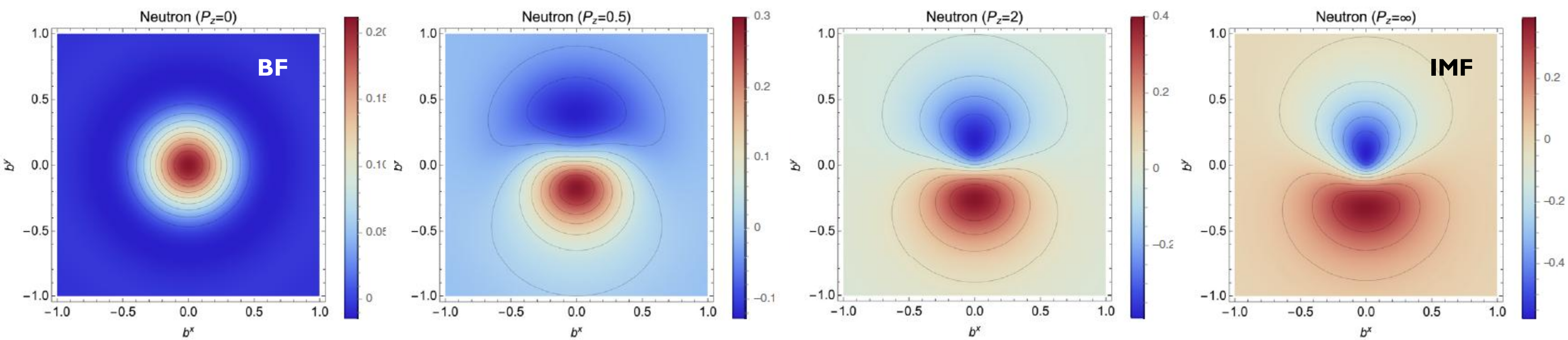
[C.L., PRL125 (2020) 232002]

EF charge distributions (transverse polarization)

Proton $\vec{S} = \frac{\hbar}{2}\vec{e}_x$



Neutron $\vec{S} = \frac{\hbar}{2}\vec{e}_x$



[Kim, Kim, PRD104 (2021) 7, 074003]